Geometric growing patterns
What’s the rule?

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We explore several examples of the way in which to use geometrical growing patterns. Examples are situated within interesting contexts to elicit algebraic generalisations by working through a five-phase analysis process.

In a Foundation year class, one student’s spontaneous response when introduced to Figure 2 below was “It looks like stairs”. This indicates that even from a young age, students demonstrate an awareness of geometric growing patterns. However, it is essential that this natural curiosity and nascent potential is capitalised and that students are challenged in a meaningful and developmentally appropriate manner.

Within the *Australian Curriculum* strand of Number and Algebra, and specifically the sub-strand of Patterns and Algebra, it is recommended that students in Year 6 “continue and create sequences involving whole numbers, fractions and decimals” and “describe the rule used to create the sequence” (Australian Curriculum, Assessment and Reporting Authority [ACARA], 2014). While teachers may address this objective through the use of number sequences (for example: 1, 4, 9, 16 — What is the rule?), using geometric growing patterns facilitates a problem-solving approach that has the potential to increase student challenge and engagement.

What is a geometric growing pattern?

While within a geometric repeating pattern, there is an identifiable core which is made up of objects that repeat in a predictable manner (see Figure 1), a geometric growing pattern (also called visual or pictorial growing patterns in other curricula) “is a pattern that is made from a sequence of figures [or objects] that change from one term to the next in a predictable way” (Billings, Tiedt & Slater, 2007, p. 303. See Figure 2).

Figure 1. Geometric repeating pattern.

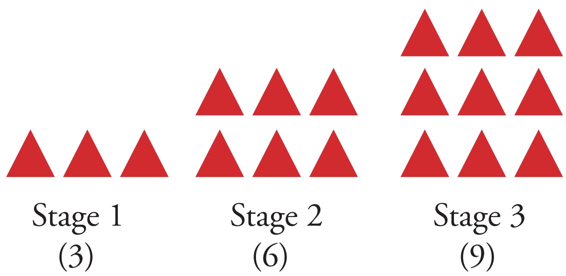


Figure 2. Geometric growing pattern.

That is, within the repeating geometric pattern shown in Figure 1, the core, consisting of a string of objects ‘triangle, triangle, rectangle’, is repeated over and over. For the growing pattern presented (Figure 2), each stage changes in a predictable manner, that is, a row of three triangles is added to each stage in order to make the next stage.

Potential use of geometric growing patterns

Research indicates that when teaching growing patterns there should not be a ‘one size fits all’ approach. Instead it is necessary to have evolving and appropriately differentiated expectations for different class groups to ensure that all students reap the benefit of engaging with geometric growing patterns from Foundation through to Year 6 (Billings et al., 2007; Friel & Markworth, 2009). As early as the Foundation year, students demonstrate a capacity to explore geometric growing patterns. Activities include copying, continuing and creating geometric growing patterns through hands-on manipulation of concrete objects (Hourigan, McMahon & Leavy, 2011). For older students, geometric growing patterns can provide a context for the development of algebraic thinking, and in particular the ability to analyse, generalise and represent relationships (Billings et al., 2007; Markworth, 2012). We run the risk, however, that if a focus is placed on the numerical data within geometric growing patterns, for example the number of objects at each stage of the growing pattern (3, 6, 9) in Figure 2, that students will focus on the recursive relationship or the change between consecutive terms or stages (Markworth, 2012). In the case of Figure 2, the recursive rule is “add 3 triangles”. While this ‘rule’ will facilitate students to find consecutive stages of a growing pattern, this approach is limiting given that in order to find the number of objects in stage 62 in the growing pattern, it is necessary to know what stage 61 looks like and add 3 triangles to it.

Students should be challenged to move beyond discovering the recursive rule and instead uncover the generalised rule for the growing pattern. To facilitate this focus, the physical construction of the pattern acts as a powerful tool in that process (Billings et al., 2007). The focus is placed on the link between the stage number and the characteristics of the pattern to find a generalised rule that explains the way in which the pattern is changing (Billings et al., 2007; Markworth, 2012). A generalised rule facilitates the discovery of the nature of any stage of a pattern.

Phases of growing pattern analysis

The approach taken to pattern analysis during the sequence of instruction presented in this paper was adapted from Billings et al.’s (2007) and Friel and Mackworth’s (2009) research on students’ processes when analysing geometric growing patterns. The phases of analysis which were implemented were developmental in nature, moving from covariational analysis of change (i.e., change from one stage to the next) to correspondence analysis of change (i.e. the relationship between the stages and characteristics of the pattern at the respective stages). The phases of analysis are:

1. Observe the growing pattern.
2. Extend the growing pattern (using previous stages).
3. Analyse the nature of the relationship.
   (a) Identify what stays the same for every stage of the growing pattern.
   (b) Identify what changes for every stage of the growing pattern (make a table if appropriate).
   (c) Identify the relationship between the stage number and what changes.
4. Predict for later stages of the growing pattern; for example, stage 50 or stage 100.
5. Generalise—ask what is the rule that can be used to tell us any stage of the growing pattern?

While phases 1 and 2 focus on covariational analysis of change, the emphasis from phase 3 through to phase 5 is on correspondence analysis of change. Each of the phases of the teaching progression will be illustrated fully through a variety of geometric growing patterns.

The context of the study

As part of Japanese lesson study research (Fernandez & Yoshida 2004; Lewis & Tsuchida 1998; Stigler & Hiebert 1999), we engaged in inquiry-based research, the focus of which was to promote the development of 5th grade students’ (aged 10–12) understandings of geometric growing patterns. Although a number of variations of lesson study exist, our structure involved the design of lessons which were taught to a class of primary school students, observations made on the teaching and on student responses, modifications to the lesson made, and the revised lessons taught once again to another group of comparably -aged students. This revised lesson was observed and further revisions made to the lesson design. We have had previous experience using the Lesson Study structure as a guiding framework in our classroom based research focusing on the teaching and learning of algebra (Leavy, Hourigan & McMahon, 2013), probability (Leavy &

Hourigan, 2014), geometry (Hourigan & Leavy, 2015; Leavy & Hourigan, 2015) and statistics (Leavy, 2010; Leavy, 2015; Hourigan & Leavy, forthcoming 2016).

In the study described here, two mathematics educators worked with five pre-service primary teachers to design a sequence of instruction focusing on geometric growing patterns for 10–12 year olds. By a sequence of instruction, we mean 2–3 mathematics lessons that focus on a specific mathematics concept. In this paper, we report on the contexts which were used to motivate students to develop their algebraic understanding. In addition to reporting the focus of each analysis phase, this paper focuses on the role of the teacher and the response of pupils when engaging with geometric growing patterns.

Launching the instruction: A context

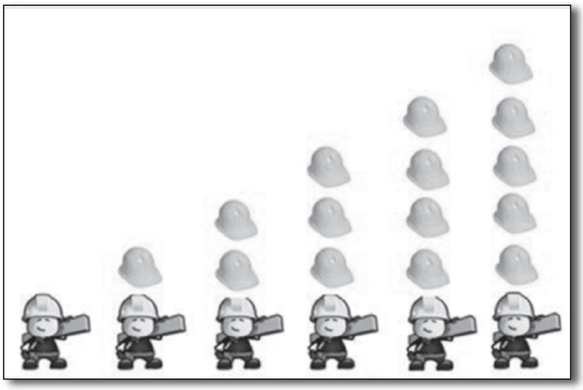
A context of 'building a skyscraper' was used for the duration of the instruction facilitating a common theme running through all of the patterns explored during the lesson. The teacher told the students that she had read a newspaper article about the building of a skyscraper. The class were encouraged to share their understandings of the characteristics of a skyscraper and photographs of a variety of skyscrapers were displayed to stimulate discussion.

Growing pattern 1: The builder and his hard hats

As research has found that students find it easier to analyse geometric growing patterns that look like recognisable objects (Billings et al., 2007), the initial activity was created to be novel and accessible (see Figure 3).

1. Observe the growing pattern

Students were presented with a geometric growing pattern of a builder (See figure 3) with position cards underneath each stage of the pattern. The teacher encouraged the students to examine the pattern by simply saying, "What do you notice?". Students were encouraged to share their observations with their partner and in turn to report back to the teacher through class discussion.



Stage 1 Stage 2 Stage 3 Stage 4 Stage 5 Stage 6

Figure 3. Growing pattern 1: The builder and his hard hats.

While initially responses such as Conor's were generic, the majority of students focused on the recursive nature of the pattern:

Conor: We see a builder and hats
Roisin: The builder has one more hat each time
Joe: It goes up like a stairs

2. Extend the growing pattern

Students were then required to extend the growing pattern. The teacher asked, "If I asked all of you to draw stage 7, what would you draw? Why?" The students' responses (e.g., "Add one hat") offered an insight into how they were 'seeing' the growing pattern and demonstrated that they recognised that the height was increasing steadily in incremental stages. In other words, the observation that stage 7 would have one additional hat indicated that they were focusing on the change from one stage to the next (covariatonal analysis of change).

3. Analyse the nature of the relationship

In order to move students towards a general rule for the growing pattern, specific questions were used to focus their observations. Students were invited to describe each of the stages through questions such as "Who can describe stage 1 for me?". As the students were initially unsure of what was required of them, some prompting was necessary through the use of questions such as: "How many hats? How many builders?" This activity drew attention to the fact that at stage 1 the builder had 1 hat; at stage 2 he had 2 hats; stage 3 he had 3 hats; and so on.

Developing from this, the teacher asked the class: "What is staying the same at each stage? What can you see at each stage that isn't changing at all?" A common response from students was "The builder isn't changing, there is always one builder." Then the teacher asked, " What is changing in each stage?" provoking the common response from students that "The number of hats the builder is wearing is changing; they keep getting bigger."

4. Predict

In order to challenge students to consider more general characteristics of the growing pattern, they were asked to predict for larger stages. At first they were asked to predict stage 10 and then stage 50. The teacher asked: "Can you describe what stage 50 in this pattern would look like?" This approach discourages the use of a recursive strategy where the focus is on adding on one more hat at each stage, thus promoting students to look for a relationship between the stage number and the number of objects in each stage of the pattern. Students were encouraged to verbalise their thoughts through questions such as "How did you work this out?"

5. Generalise

The focus of the next phase was to support students in creating a general rule to describe the nature of the growing pattern. Students were asked to consider how they would describe the pattern to someone who had not seen it (Friel & Markworth, 2009).

Teacher: Imagine I took the picture down. If someone came into the room and couldn't see the picture, how would you describe to them what was happening? How would you explain it to them?

After some silence, students shared their efforts. Initial descriptions focused on the overall appearance of the growing pattern such as "Each stage is getting higher by each hat of the builder". The teacher responded by engaging in prompting and probing questions such as: "Each stage has what? What does stage 3 have? Stage5? Stage 15? Do you see any kind of relationship there?" This resulted in clarification as indicated by a student's response: "The number of the stage is the same as the number of hats." This led to the suggestion that the rule was "One builder plus the stage number of hats". The students were made aware that while words can be used, sometimes they are not very efficient. The teacher explained:

Teacher: It would take a long time to write all of that down wouldn't it? Let's see if we can shorten it down as much as possible so 'builder + ______ hats'. What will I put in here ? (Pointing to the blank space.)

It was intended that the students would be introduced to the notion of using a variable. Further prompting and discussion focused on how we sometimes use a letter to stand for something, for example we might use h to stand for the number of hours homework. It was highlighted that it is not necessary to use the first letter of the word to which a student suggested using k. Both classes were comfortable to use a variable such as x, where x represents the stage number. Therefore the rule the class created was builder + x hats (where x = stage number). On reflection, we felt there was an opportunity to allow pupils to use this 'rule' to solve a range of problems and see the benefits of having a generalised rule for the growing pattern.

Growing pattern 2: Skyscraper problem

The students were then presented with a problem:

Teacher: We are going to build a skyscraper. Our builder is going to order all the windows for the skyscraper. The site our skyscraper will be built on is very narrow. As a result, we have to build a narrow and tall skyscraper. Each floor will be square in shape. Our builder wants just one window on each side/wall of the skyscraper and he always wants a window on the roof—like a sunroof—on the top.

We made a conscious decision to refer explicitly to the shape of each floor in order to provide some rationale for using the same number of windows on each side/wall. As an extension or in the case of differentiated instruction, it would be possible to increase the complexity of the problem by considering alternative shaped floors or by varying the number of windows on the sides/walls of the building.

1. Observe the growing pattern

This time, rather than being presented with a pattern, the students had to use the information to predict the number of windows on skyscrapers of various storeys. Students were given cubes to represent storeys of the building and stickers to represent windows. After making and justifying their predictions students used the materials to create the buildings and verify the accuracy of their predictions. Initially the students were asked to work out how many windows on a one-storey skyscraper (Answer: 5). Then they worked in pairs to double-check (a sticker was placed on each cube to represent the windows); (see Figure 4).

Figure 4. Students creating, observing and extending the skyscraper growing pattern.

2. Extend the growing pattern

The approach described above was taken for two, three and four storeys. Students recorded and presented a justification for their prediction of the total number of windows and then created a model of the building using cubes and used this model to check and confirm the actual number of windows (see Figure 4). Students were encouraged to keep the model of each stage and make a new tower/skyscraper for each subsequent stage. Throughout, multiple ways of seeing and counting were encouraged and valued (Markworth, 2012). It was interesting to observe the range of strategies used to identify the number of windows represented on a four-storey building (i.e., 13 windows):

Student 1: There are 9 there [pointing to the 2 storey building] and you add 4 because the new storey will have 4 more windows.
Student 2: 4 + 4 + 4 + 1 [pointing to the 3 floors which had been attached].
Student 3: I multiplied to get my answer. There are 3 blocks with 4 windows so that is 12 and I added on the sunroof which makes 13.
Student 4: I went 3, 3, 3, 3, 1 [pointing down the sides of the 3 floors] that is 3, 6, 9, 12 +1 on the top.

After the students made their predictions, the teacher queried "Why is it not fifteen windows when there are three floors?" In response one student said "Because those ones in the middle don't count because you can't actually see them so there's no sunroof." This student was referring to the fact that all floors expect for the top floor have four windows as there was no sunroof.

3. Analyse the nature of the relationship

At this point, each pair of students had a model for skyscrapers with one, two, three and four storeys on their desks. Initial discussion focused on the parts of the pattern that always stayed the same for each stage. The students were quick to identify that on each skyscraper there was always one window on top. The teacher tested this proposal with questions such as: "If the building is 25 storeys high, will it still be the same?" Students were confident that regardless of the number of storeys there was always one window on the top of the building.

When asked, students had no difficulties identifying what was changing in the pattern; for example, "the number of windows on the side of the building". On this occasion, a three-column table (see Table 1) was used to analyse the characteristics of the pattern further (Friel & Markworth, 2009; Markworth, 2012). The headings used for each of the three columns of this table were: number of storeys, windows on the sunroof and windows around the sides.

On reflection, these headings should have been more explicit (see Figure 5). The teacher engaged students in discussion of the characteristics of the pattern and recorded the findings on the three-column table for each stage of the pattern (see Figure 5 and Table 1):

Teacher: So in our first storey skyscraper, how many windows go on top and how many go on the sides?

Student 1: One on top and four on sides.
Teacher: Look at your two-storey skyscraper, what will I record for this?
Student 2: One on top and eight on the sides.

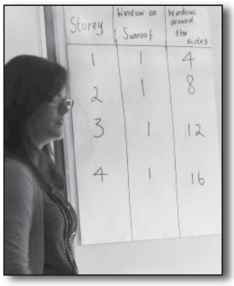


Figure 5. A teacher using a three-column table to focus on characteristics of the growing pattern.

This activity facilitated the teacher to focus on the relationship between the number of storeys (stage number) and the number of windows on the side of the building (what was changing).

Teacher: Let's look at our table here. Does anyone notice a relationship between the number of storeys and the number of windows for the sides? Can anyone see a pattern?
Student 1: The number of sides is like the four times tables (number facts).

This phase concluded by having students consider if this relationship worked for a five storey building.

Teacher: If I have five storeys, what will I put here? [pointing to the 2nd column which refers to the sunroof] and how many on the sides [pointing to the third column which refers to the sides]?

4. Predict

In accordance with the sequence of instruction adopted, students were then asked to predict for a larger stage in the pattern. It is important to select a stage that would not be possible to build recursively: "Imagine we want a building that is 100 storeys high; how many windows would we need?" Support was provided through prompts such as: "Think about how many windows will go on the top and how many on the sides. Think about what we have just discovered."

Students shared their answers and provided justifications. Where pupils struggled, they were encouraged to consider smaller stages initially for example six storeys, seven storeys and then moving to considering 10 storeys. A focus was placed on sharing their thinking and approaches.

Table 1: Headings and contents of the three-column table.

Number of storeys	Number of windows on sunroof	Total number of windows around sides of building
1	1	4
2	1	8
3	1	12
4	1	16

During the pattern extension work (phase 2 of the pattern analysis) some of the students were starting to generalise and able to predict for a large number. In the brief dialogue below two students are considering the number of windows if the skyscraper had 50 storeys.

Teacher: How many windows if there were 50 storeys?
Student 1: Two hundred. No! There would be 201.
Teacher: Tell me how you know this.
Student 2: There are four windows on each side and there were 50 storeys and one on top. That makes 201.

5. Generalise

Students were given the opportunity to work in pairs to create a rule for the pattern. They were given a rationale for this:

> It was decided that this type of skyscraper would be built in this city. However the builder who got the job was informed that it would have at least 100 storeys, the owner had not decided exactly how many. The builder was anxious as he wanted to order the windows straight away as it had to be built as quickly as possible. Rather than having to count the number of windows storey by storey when he got the information, he decided he needed to come up with a rule to figure out,

no matter the number of storeys, how many windows he would need to order for the skyscraper. Can you work out the rule?

Students were encouraged to create a verbal or symbolic rule. The teacher emphasised how important it was for them to test their rule to see if it always worked. Students were initially encouraged to focus on words. They were reminded of the process for the builder's hats. They shared their rules for example the number of windows is 4 times the number of storeys plus one for the sunroof. Then the focus moved to creating a symbolic rule. As the teacher circulated, she provided support as necessary: "If *n* means the number of storeys, how would you describe the number of windows?" Students shared their rules such as "4s +1; where s is the number of storeys" and the class responded with thumbs up (agree) or thumbs down (disagree).

Growing pattern 3: Dinner tables problem

At this stage of the instruction, the class were given the opportunity to use their learning to discover the rule for a new growing pattern. The students were introduced to the problem.

Teacher: The builders wanted to do something special with the top floor because of the sun-roof so they decided to turn it into a restaurant.The tables they selected for the restaurant are rectangles that can fit two people at the long side and one person at the short side (see Figure 6).

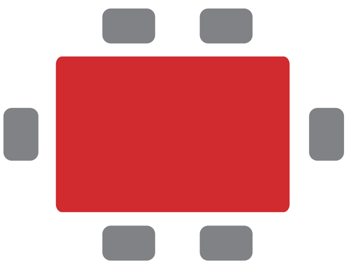


Figure 6. Chairs around one rectangular table.

Figure 7. Image of rectangular tables pushed together at the short side.

After exploring the total number of chairs which would fit around one table (see Figure 6), the teacher informed the students that all the dinner tables would be pushed together: "We want to get the most out of our space so we are going to push the tables together so there are no gaps between them."

The class was presented with an image of two tables joined together at the short sides (see Figure 7). It is possible at this stage or as a subsequent activity to take a more open-ended approach and allow students to join the tables in a variety of different ways.

The teacher gave the students the task of identifying how many chairs would be needed for two tables. They were encouraged to predict prior to using models to check. A common mistake was to use direct multiplicative reasoning and presume that if one table fitted six chairs, then two tables would fit 12 chairs. The students worked in pairs and created models using rectangular pieces of card to represent tables and circular counters to represent chairs (see Figure 8). These models were used to create and extend the pattern (phases 1 and 2 of the pattern analysis). Students were invited to share their strategies and also to explain to their peers why 10 rather than 12 chairs could fit around two tables. A common response was: "...because when they are pushed together the two chairs in the middle have to go."

Figure 8. Students creating, observing and extending the table growing pattern.

Then students were encouraged to independently explore stages 3 and 4 of the pattern; i.e., number of chairs at three tables and four tables. It is interesting that at this point while some predictions relied on the fact that "it is going up in fours", the thinking of others was more sophisticated. One example is a student predicting the number of chairs at two tables stated "four on the top, four on the bottom and two at the end".

The focus then moved to phase 3 of the pattern analysis (what stayed the same and what changed). Students had little difficulty identifying that there are always two chairs at the ends. They also recognised that the "number of chairs at the sides" changed from stage to stage. A three-column table was completed to record these features in an effort to identify the relationship between the number of tables (stage number) and what changed. Again students noticed that the "number at the sides is going up in fours" or "It's like the four times tables again" and in turn worked towards the rule (orally or equation): "Multiply the number of tables by four and add two" ($4p + 2$; where p is the number of tables). The class were asked to check if the rule worked for three tables and to use it to find the number of chairs for 11 tables also.

Then in order to test the students' levels of understanding, the teacher provided a more challenging task: "If 34 people could sit around, how many tables would be pushed together?" After pair work, pupils reported back, sharing their solutions (eight tables) and strategies:

Student 1: Eight by 4 is 32 and 2 at the end.
Student 2: Four into 34 = 8 and 2 left for the end.
Student 3: We subtracted 2 first of all and then divided by 4.

As an extension activity, the students were presented with a 'trapezium table problem' through the continued use of the context:

> Let's imagine our builder is very fussy and he doesn't want rectangular tables at all. Instead he wants trapezium shaped tables—just to be different! So there's one short side and one long side and two end sides. Two people can fit on the long side; one can fit on the short side and one on each of the end sides. So as you can see, five people can fit around one table (see Figure 9). Now these tables are also pushed together—to tessellate—so the second table needs to be turned around to fit (See figure 10). How many people can fit around two tables?

While the teacher continued to act in the role of guide, it was intended that students would be given more independence in solving this problem. Students were encouraged to predict the number of chairs at each stage (for two tables, three tables and four tables). They were also given materials to help them to verify their predictions for the initial stages (see Figure 11). They worked through the five phases of the analysis (as outlined above) to uncover the rule for this pattern in their pairs (Rule: $3n + 2$, where n = number of tables). Opportunities were taken throughout for appropriate questioning, sharing of strategies as well as discussion.

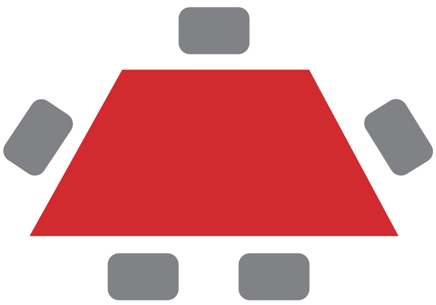


Figure 9. Chairs around 1 trapezium table.

Figure 10. Tessellating trapezium tables.

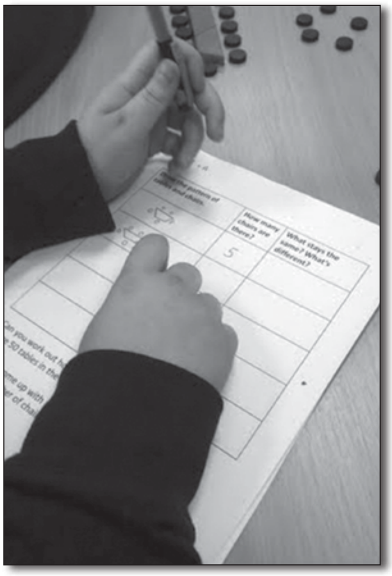


Figure 11. Students working on the trapezium table problem.

Reflections

For some time, the use of geometric growing patterns has been advocated internationally as a tool for developing students' algebraic skills and in particular their ability to generalise, i.e., create rules (Markworth, 2012). While the *Australian Curriculum* (ACARA, 2014) does not make any explicit reference to this approach, this study exhibits that the demonstrated approach to geometric growing patterns has great potential to capture students' interest, as well as providing appropriate support and challenge which in turn results in success for all. This research illustrates that the provision of an interesting context provides specific affordance that support developing understandings of geometric growing patterns. Such contexts, as in the skyscraper problem, provide physical manifestations and referents that structure and scaffold emerging understandings of the geometric growing patterns. There is much potential for adjustments and changes to be made to the activities described in this paper; for example teachers may decide to provide more support or challenge as appropriate. Overall, the use of the five phases of analysis structure facilitated students to move from covariational analysis of change to correspondence analysis of change. Situating the activities within a variety of related contexts provides coherence, motivation and support for students to develop the appropriate conceptual understanding. Most of all it has the potential to ease them into the world of algebraic thinking!

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